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Robust vibration control of dynamical systems based on the derivative of the state

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Summary Vibrations may be undesirable in dynamical systems for several reasons. They may affect the security, such as vibrations in primary cycle parts of (nuclear) power plants. They may decrease the quality and functionality of products, such as those manufactured by machine tools. And they may lower the comfort, such as vibrations in car wheel suspension systems or in power trains of cars. One possibility to attenuate these vibrations is by employing active suspension elements. Mounted at appropriate places inside the systems or with respect to their environment, they are able to interchange or dissipate kinetic and potential energy in an effective way with moderate control effort. Their effectiveness depends greatly on the control scheme applied to change damping and stiffness characteristics of the suspension elements. The control schemes, however, very often need information on the state variables involved in the mathematical modeling. On the other hand, it is mostly the acceleration or speed of certain parts that can be sensed reasonably and measured with sufficient accuracy. We propose here a control scheme which is solely based on the derivative of the state variables, provided that active suspension elements or actuators with the above-mentioned properties may be employed within the system. Furthermore, we only use control actions within a discrete set of possible values, which aids the real-time implementation of the designed control algorithms. And, last but not least, the number of control inputs (actuators) may be arbitrary, that is, the system may be mismatched. The scheme is based on the Lyapunov stability theory, which involves discontinuities of the Lyapunov function candidates along trajectories of the state derivative. The effectiveness and behavior of the control scheme is demonstrated on a two-DOF model of an active car seat suspension in order to enhance the driving comfort.

Keywords Vibration, Attenuation, Robust control, Constrained control, Lyapunov stability

1 Introduction

The class of systems which we shall take into consideration may be described by a dynamical system with a finite number of degrees of freedom (DOF). The structure has to contain "active" suspension elements. We call suspension or coupling elements "active" if they are adjustable with respect to their stiffness and damping behavior. Based on that model, we assume that a control action is related to a change in these properties. The mathematical description of this kind of structures is assumed to be of the form

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} + \mathbf{e}, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \ . \tag{1}$$

The linear part of the mathematical model is defined by the constant and stable matrix $\mathbf{A} \in \mathbb{R}^{n,n}$, where $n \in \mathbb{N}$ denotes the state-space dimension. The control input matrix

$$\mathbf{B}(\mathbf{x}) := \mathbf{B}_1(\mathbf{x}) + \mathbf{B}_2 \in \mathbb{R}^{n,m}$$

may contain some constant part $\mathbf{B}_2 \in \mathbb{R}^{n,m}$ and some part $\mathbf{B}_1(\mathbf{x}) \in \mathbb{R}^{n,m}$, which is linear with respect to $\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ represents the n state variables; $\mathbf{u} \in \mathbb{R}^m$ represents the m control variables. Furthermore, we will assume that only $\mathbf{y} := \dot{\mathbf{x}}$ is detectable via some appropriate measurement device. The control variables have to be taken from the set

$$\mathcal{U} := \{\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^m \mid p_j(\mathbf{y}) \in \{u_{j,\min}, 0, u_{j,\max}\} \quad \forall\ j = 1, \ldots, m\}, \tag{2}$$

where $\mathbf{p}$ is supposed to be piecewise continuous with respect to the measured and/or observed variable $\mathbf{y}$. Without loss of generality and for the sake of convenience, we may assume that $u_{j,\min} = -1$ and $u_{j,\max} = +1$. The particular choice of control action $p_j(\mathbf{y})$ which either takes the minimum value $u_{j,\min}$, 0 or the maximum value $u_{j,\max}$ almost everywhere, is motivated by the control design presented in [5]. All uncertainties and nonlinearities of the system are modeled by some appropriate, at least piecewise differentiable function $\mathbf{e}(t)$. Its time derivative is assumed to be bounded, that is,

$$\|\dot{\mathbf{e}}(t)\| \leq \varepsilon \quad \forall\ t \in \mathbb{R}_+ \ . \tag{3}$$

holds for some properly chosen $\varepsilon \in \mathbb{R}_+$.

2
Control design

For the controller design, we ask for a feedback controller $\mathbf{u} := \mathbf{p}(\mathbf{y})$ which drives the measured and/or observed variable $\mathbf{y}$ towards a ball of ultimate boundedness

$$\mathcal{B}_\rho := \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq \rho\}$$

for some properly chosen real number $\rho > 0$. As a first step towards this objective, the time derivative of Eq. (1) leads to

$$\ddot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{x}} + \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{B}(\mathbf{x})\mathbf{u}] + \dot{\mathbf{e}} \ . \tag{4}$$

Taking into account

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{B}(\mathbf{x})] = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{B}_1(\mathbf{x}) + \mathbf{B}_2] = \mathbf{B}_1(\dot{\mathbf{x}}) \ , \tag{5}$$

due to the linearity of $\mathbf{B}_1$ with respect to $\mathbf{x}$ leads to

$$\ddot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}_1(\dot{\mathbf{x}})\mathbf{u} + \mathbf{B}(\mathbf{x})\dot{\mathbf{u}} + \dot{\mathbf{e}} \ . \tag{6}$$

Since the components u_j of $\mathbf{u}$ take the constant values $u_{j,\min}$, 0 or $u_{j,\max}$ almost everywhere, that is on open subsets of $\mathbb{R}^n$, except on measure-zero sets of $\mathbb{R}^n$, the time derivative of $\mathbf{u} = \mathbf{p}(\mathbf{y})$ along any solution $t \mapsto \mathbf{y}(t)$ of Eq. (6) is zero for all $\mathbf{p}$ belonging to $\mathcal{U}$, provided chattering does not occur on the $(n-1)$-dimensional manifold

$$\Pi_j := \{\mathbf{y} \in \mathbb{R}^n \mid b_j(\mathbf{y}) = 0\}, \quad \text{for any } j \in \{1, \ldots, m\} \ , \tag{7}$$

where $b_j(\mathbf{y})$ will be defined subsequently. Hence, we obtain

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1(\mathbf{y})\mathbf{p}(\mathbf{y}) + \dot{\mathbf{e}}, \quad \mathbf{y} \in \mathbb{R}^n - \bigcup_{j=1}^{m} \Pi_j \ . \tag{8}$$

This result enables us to return to the control-design procedure introduced in [5]. There we ask for a feedback-control function $\mathbf{p}^* \in \mathcal{U}$, which – for an arbitrary but fixed positive-definite matrix $\mathbf{P} \in \mathbb{R}^{n,n}$ – minimizes the "Lyapunov derivative"

$$\mathcal{L}(\mathbf{p}) := \mathbf{y}^T\mathbf{P}[\mathbf{A}\mathbf{y} + \mathbf{B}_1(\mathbf{y})\mathbf{p} + \dot{\mathbf{e}}] = \mathbf{y}^T\mathbf{P}\left[\mathbf{A}\mathbf{y} + \sum_{j=1}^{m} p_j\mathbf{B}_1(\mathbf{y})\mathbf{i}_j + \dot{\mathbf{e}}\right] , \tag{9}$$

with respect to $\mathbf{p} \in \mathcal{U}$ for every $(\mathbf{y}, t) \in (\mathbb{R}^n - \bigcup_{j=1}^{m} \Pi_j) \times \mathbb{R}_+$. Here, $\mathbf{i}_j$ denotes the unit vector with $\mathbf{i}_i^T\mathbf{i}_j = 0$ for $i \neq j$. In that case, the time derivative of the Lyapunov function candidate

$$V(\mathbf{y}(t)) := \tfrac{1}{2}\, \mathbf{y}(t)^T\mathbf{P}\mathbf{y}(t) \tag{10}$$

will be as small as possible for any response $t \mapsto \mathbf{y}(t)$, admissible uncertainty $\dot{\mathbf{e}}$ and time t, and for all admissible choices of control $\mathbf{p}(\mathbf{y}(t))$. Eq. (9) can be written as

$$\mathcal{L}(\mathbf{p}(\mathbf{y})) = a(\mathbf{y}) + \sum_{j=1}^{m} p_j(\mathbf{y})b_j(\mathbf{y}) + c(\mathbf{y}, t) , \tag{11}$$

with

$$a(\mathbf{y}) := -\tfrac{1}{2}\mathbf{y}^T\mathbf{Q}\mathbf{y}, \quad \text{where } \mathbf{Q} := -(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P}) ,$$
$$b_j(\mathbf{y}) := \mathbf{y}^T\mathbf{P}\mathbf{B}_1(\mathbf{y})\mathbf{i}_j, \quad \text{where } \mathbf{i}_j^T\mathbf{i}_k = \delta_{jk} \tag{12}$$

and

$$c(\mathbf{y}, t) := \mathbf{y}^T\mathbf{P}\dot{\mathbf{e}}(t) . \tag{13}$$

Then, using the normalized control space $\mathcal{U}$, we obtain

$$p_j^*(\mathbf{y}) = \begin{cases} -1 & \text{if } b_j(\mathbf{y}) > 0, \\ +1 & \text{if } b_j(\mathbf{y}) < 0 . \end{cases} \tag{14}$$

The performance of the controller may be enhanced additionally if we choose $\mathbf{P}$ appropriately. The smaller the Lyapunov derivative, the stronger the "tendency to the origin" of $t \mapsto \mathbf{y}(t)$. Our objective in that case would be to strive for a highly negative value $a(\mathbf{y})$ in Eq. (11). That, on the other hand, can be done by choosing a suitable positive-definite $\mathbf{Q}$ and solving the algebraic Lyapunov equation

$$\mathbf{Q} = -(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P}) \tag{15}$$

for $\mathbf{P}$. Since $\mathbf{A}$ is assumed to be stable, the matrix $\mathbf{P}$ is positive-definite.

3
Stability

Using the control scheme developed in Sec. 2, it is possible to determine a sufficient condition between parameters of the system and control design parameters for ultimate boundedness with respect to some ball $\mathcal{B}_\rho$ of any response $t \mapsto \mathbf{y}(t)$. In order to analyze this situation, we employ, for any given positive-definite matrix $\mathbf{P} \in \mathbb{R}^{n,n}$, the controller $\mathbf{p}^*(\mathbf{y})$, cf. Eq. (13) in its corresponding Lyapunov derivative[1]. This leads to

[1] It is possible to deal with uncertain $\mathbf{y}$ measurements in the manner discussed in [12], here setting $p_j(\mathbf{y}) = 0$ in an appropriate neighborhood of Π_j for any $j \in \{1, \ldots, n\}$. In [12], a so-called fuzzy controller is used making the controller a continuous function of $\mathbf{y}$, thereby also precluding chattering. However, this can not be done here in view of the discrete-valuedness of admissible control.

$$\begin{aligned}\mathcal{L}(\mathbf{p}^*(\mathbf{y})) &= -\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{y} + \mathbf{y}^{\mathrm{T}}\mathbf{P}\mathbf{B}_1(\mathbf{y})\mathbf{p}^*(\mathbf{y}) + \mathbf{y}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} \\ &= -\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{y} - \mathbf{y}^{\mathrm{T}}\mathbf{P}\mathbf{B}_1(\mathbf{y})\left(\sum_{j=1}^{m} p_j^*(\mathbf{y})\mathbf{i}_j\right) + \mathbf{y}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} \\ &= -\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{y} - \sum_{j=1}^{m} |b_j(\mathbf{y})| + \mathbf{y}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} \ . \end{aligned} \tag{16}$$

Since $\mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{y}$ is a positive-definite quadratic form, it is bounded by the minimum and maximum eigenvalues $\lambda_{\min}(\mathbf{Q})$ and $\lambda_{\max}(\mathbf{Q})$ of $\mathbf{Q}$. That is,

$$\mathcal{L}(\mathbf{p}^*(\mathbf{y})) \leq -\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{y}\|^2 - \sum_{j=1}^{m} |b_j(\mathbf{y})| + \|\mathbf{y}\| \ \|\mathbf{P}\| \ \|\dot{\mathbf{e}}\| \ . \tag{17}$$

Neglecting the second term, employing inequality (3) and using the maximum eigenvalue $\lambda_{\max}(\mathbf{P})$ of $\mathbf{P}$ one gets

$$\mathcal{L}(\mathbf{p}^*(\mathbf{y})) \leq -\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{y}\|^2 + \varepsilon\lambda_{\max}(\mathbf{P})\|\mathbf{y}\| \ . \tag{18}$$

Thus, we obtain

$$\mathcal{L}(\mathbf{p}^*(\mathbf{y})) < 0 \quad \forall \ \|\mathbf{y}\| > r := 2\,\varepsilon\,\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{Q})} \ . \tag{19}$$

The radius ρ of the ball of ultimate boundedness $\mathcal{B}_\rho := \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq \rho\}$ is therefore determined by

$$\rho = r\sqrt{\frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})}} \ . \tag{20}$$

It should be noted that $\mathcal{B}_\rho$, with radius given in (20), is the ball of ultimate boundedness for control actions $u_j = 0\,\forall j$, since the terms $\sum_{j=1}^{m}|b_j(\mathbf{y})|$ were neglected in (18). These terms reduce the right-hand side of (18), except at $\mathbf{y}$ where $b_j(\mathbf{y}) = 0$ for *all* j. Thus, control $\mathbf{p}^*(\mathbf{y})$ has two effects. It decreases the rate of change of the P-norm $V(\mathbf{y}), \mathcal{L}(\mathbf{p}^*(\mathbf{y}))$, and it increases the region in which the negativity of $\mathcal{L}(\mathbf{p}^*(\mathbf{y}))$ is assured.

It should be noted that the control scheme (14) pertains to the open regions separated by switching manifolds (7). For $\mathbf{y}(t)$ on a switching manifold Π_j, the terms dependent on $\mathrm{d}\mathbf{u}/\mathrm{d}t$, which is unbounded there, lead to Dirac-delta changes in $\mathbf{y}$ and V in the ideal model under discussion.

In practice, in a neighborhood of a switching point, rapid changes in $\mathbf{y}$ and V occur due to control with "high" gain over a "short" time interval. In the stability analysis, we assumed that the rapid changes experienced by the Lyapunov function $V(\mathbf{y}(t))$ at a switch point of the control are sufficiently small compared to the decrease in $\mathcal{L}(\mathbf{p}(\mathbf{y}))$ between switch points, or even benign, so that they can be ignored. This seems to be borne out by the simulation results, but needs closer examination in general. We intend to investigate this phenomenon in subsequent research.

Finally, it should be recalled that the proposed control scheme is "optimal" with respect to the stability behavior of $\mathbf{y}(t)$ rather than $\mathbf{x}(t)$, although it may be expected that it also improves the behavior of $\mathbf{x}(t)$, as it is borne out by simulations. For many systems of interest, the accelerations of the system components can be measured directly via accelerometers.
The component velocities can then be obtained from integration which, of course, involves the initial values of the velocities. In many cases, the system starts from rest, in which case these are zero. If not, the unknown or uncertain initial velocity values introduce an offset error. However, this is still a great improvement over the situation of a control based on the state $\mathbf{x}$. That one involves two integrations, and since the measurements are still those of the component accelerations, it greatly increases the errors due to lack of knowledge or uncertainty of the

initial x. In addition, for many systems of interest, improving the behavior of accelerations and velocities is of greater importance than the behavior of displacements.

4
Example: active suspension of a car seat

Figure 1 shows a simplified model of an actively suspended seat in a car. The model consists of a car mass M and driver-plus-seat mass m. Vertical vibrations caused by a rough street may be partially attenuated by shock absorbers (stiffness k_A and damping c_A). Nonetheless, the driver may still be subjected to undesirable vibrations. These vibrations, again, can be reduced by appropriately mounted car seat suspension elements. The elastic mounts are considered to be active with the stiffness $k_S(u) = \alpha_k + \beta_k\, u$ and the damping $c_S(u) = \alpha_c + \beta_c\, u$, where u is the normalized and constrained control variable. That is, stiffness as well as damping can be varied by changing the scalar variable u. The vertical displacement $\zeta(t)$ is considered to be unknown, but possibly with a known bound. We assume that the accelerations $\ddot{\xi}$ or $\ddot{\eta}$ can be measured. The velocities $\dot{\xi}$ and $\dot{\eta}$ are either also measured, or at least estimated from their measured time derivatives. That yields the following matrices for the model according to Eq. (1):

$$\mathbf{A} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_A+\alpha_k}{M}\right) & \frac{\alpha_k}{M} & -\left(\frac{c_A+\alpha_c}{M}\right) & \frac{\alpha_c}{M} \\ \frac{\alpha_k}{m} & -\frac{\alpha_k}{m} & \frac{\alpha_c}{m} & -\frac{\alpha_c}{m} \end{bmatrix} \tag{21}$$

and

$$\mathbf{B}(\mathbf{x}) := \mathbf{B}_1\mathbf{x} + \mathbf{B}_2 \ , \tag{22}$$

where

$$\mathbf{B}_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\beta_k}{M} & \frac{\beta_k}{M} & -\frac{\beta_c}{M} & \frac{\beta_c}{M} \\ \frac{\beta_k}{m} & -\frac{\beta_k}{m} & \frac{\beta_c}{m} & \frac{\beta_c}{m} \end{bmatrix}, \quad \mathbf{B}_2 := \begin{bmatrix} 0 \\ 0 \\ -\frac{\beta_k\,\delta_S}{M} \\ \frac{\beta_k\,\delta_S}{m} \end{bmatrix}. \tag{23}$$

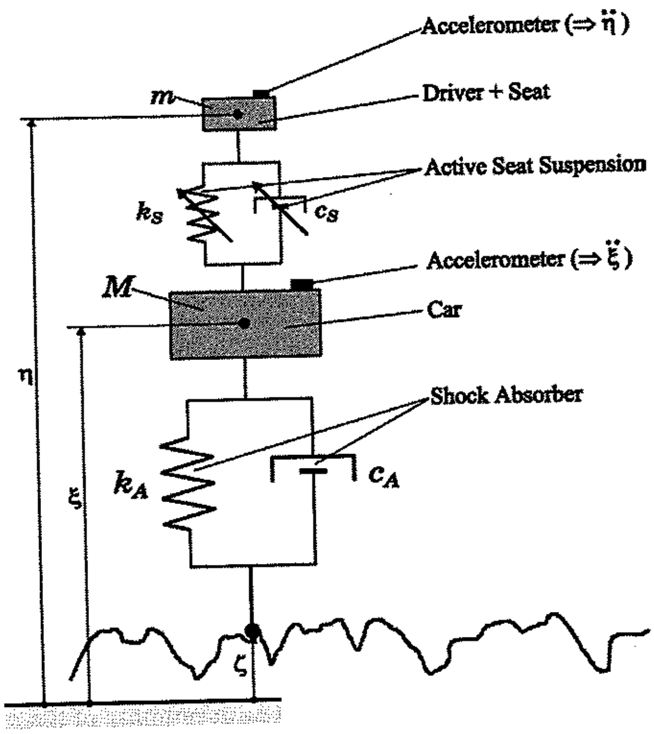


Fig. 1. Model of an actively mounted seat inside a car

The state vector is given by

$$\mathbf{x} := (\xi, \eta, \dot{\xi}, \dot{\eta})^{\mathrm{T}} \ .$$

The ground excitation leads to

$$\mathbf{e}(t) := \begin{bmatrix} 0 \\ 0 \\ \frac{c_A}{M}\dot{\zeta}(t) + \frac{k_A}{M}\zeta(t) \\ 0 \end{bmatrix} \ . \tag{24}$$

Simulations for $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are based on the original state Eq. (1) with control (14), which depends on $\mathbf{y} := \dot{\mathbf{x}}$, namely

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x})p^*(\mathbf{y}) + \mathbf{e}(t) \ . \tag{25}$$

The simulation results, cf. Figs. 2 and 3, are is based on the parameters $g = 9.81$ m/s^2 (gravitational constant), $m_C = 1500$ kg (mass of the car), $m_S = 100$ kg (mass of the seat + driver), $k_A = 4 \cdot 10^4$ N/m (shock absorber stiffness), $c_A = 4 \cdot 10^3$ Ns/m (shock absorber damping), $\alpha_k = 10^4$ N/m, $\beta_k = 5 \cdot 10^3$ N/m, $\alpha_c = 10^2$ Ns/m, $\beta_c = 5 \cdot 10^2$ Ns/m; δ_A and δ_S are chosen in

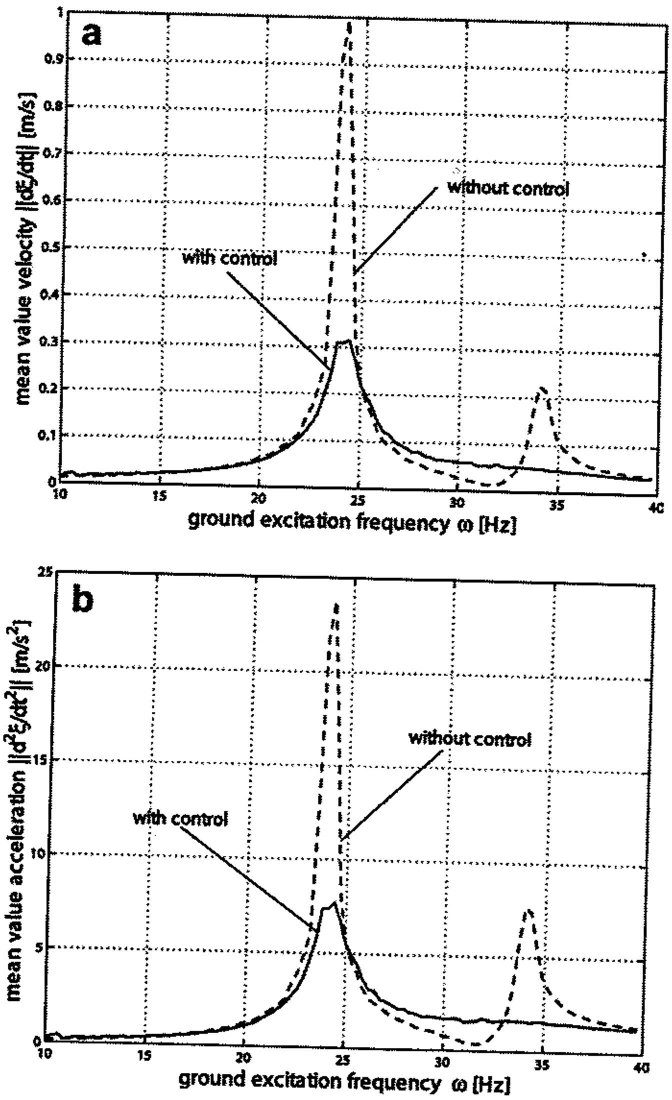


Fig. 2. **a** Mean value $\|\dot{\xi}\|$ vs. excitation frequency ω. **b** Mean value $\|\ddot{\xi}\|$ vs. excitation frequency ω

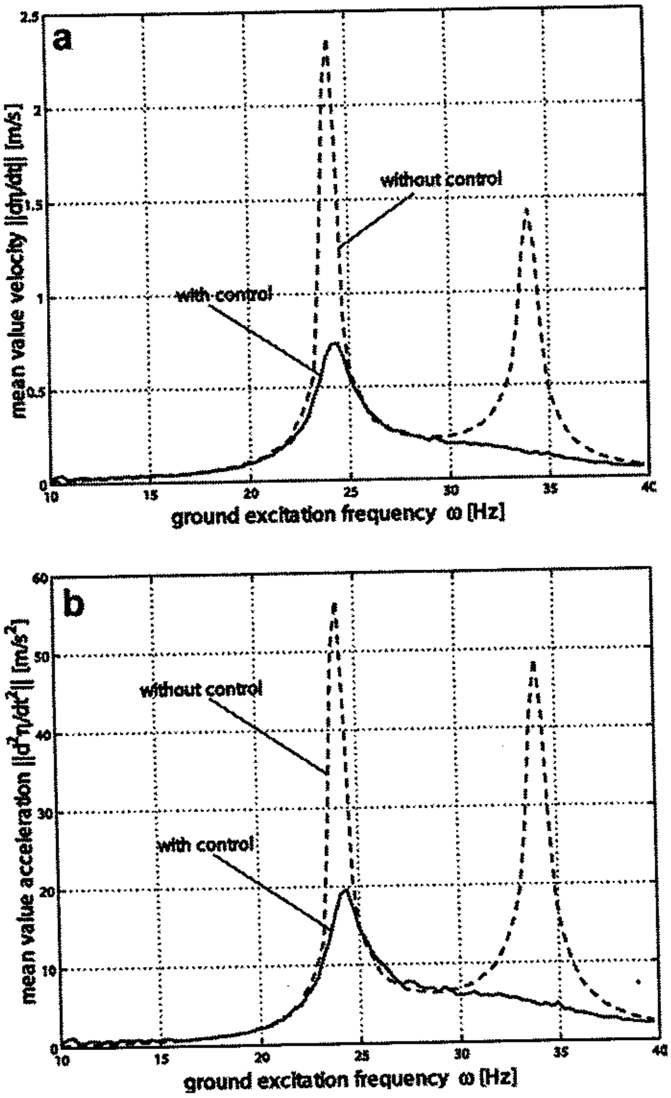


Fig. 3. a Mean value $\|\dot{\eta}\|$ vs. excitation frequency ω, b mean value $\|\ddot{\eta}\|$ vs. excitation frequency ω

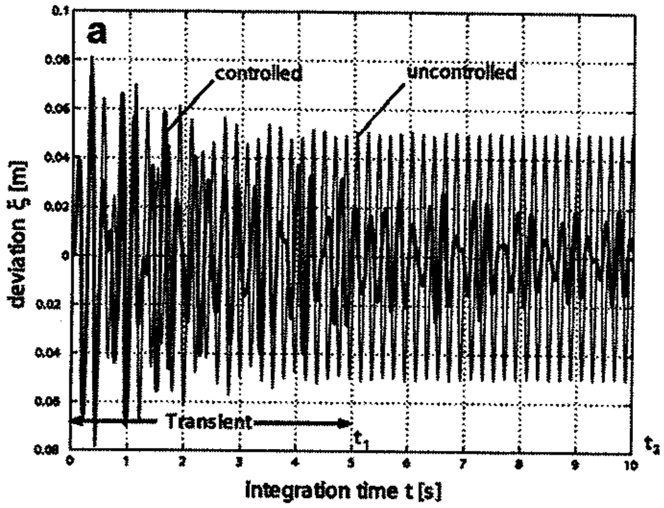
such a way, that the shock absorber elements are relaxed for $(\xi - \zeta) = \delta_A$ and the active seat suspension elements are relaxed for $(\eta - \xi) = \delta_S$. That leads to

$$\delta_S = \frac{m\,g}{\alpha_k} \quad \text{and} \quad \delta_A = \frac{(m+M)\,g}{k_A} \ . \tag{26}$$

The ground excitation is taken to be harmonic, that is $\zeta(t) := \hat{\zeta}\ \sin(\omega\ t)$, with amplitude $\hat{\zeta} := 0.03$ m and a variable excitation frequency 10 Hz $\leq \omega \leq$ 40 Hz. In order to measure the effect of the employed controller, we integrate the state equation between time $t_0 = 0$ s and $t_2 = 10$ s. After a settling time $t_1 = 5$ s, for the homogeneous parts of the state variables to be practically damped out, we compute the mean value

$$\|x_i\| := \sqrt{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |x_i(t)|^2 \mathrm{d}t} \ , \tag{27}$$

for the components x_i of $\mathbf{x}$ as functions of the excitation frequency ω. To demonstrate the efficacy of the control in enlarging the region in which negativity of $\mathcal{L}(\mathbf{p}(\mathbf{y}))$ is assured the numerical integration of (25) was started with the initial condition $\mathbf{x}(t_0 = 0) = \mathbf{0}$. The values of $\mathbf{y}(t)$ may be directly obtained from (25). Figure 2 shows the results with and without control for the variables $y_1 = \dot{\xi}$ and $y_3 = \ddot{\xi}$. Figure 3 shows the results for the variables $y_2 = \dot{\eta}$ and $y_4 = \ddot{\eta}$, again with and without control. Although there is a slight deterioration of $\dot{\xi}$ and $\ddot{\xi}$ between



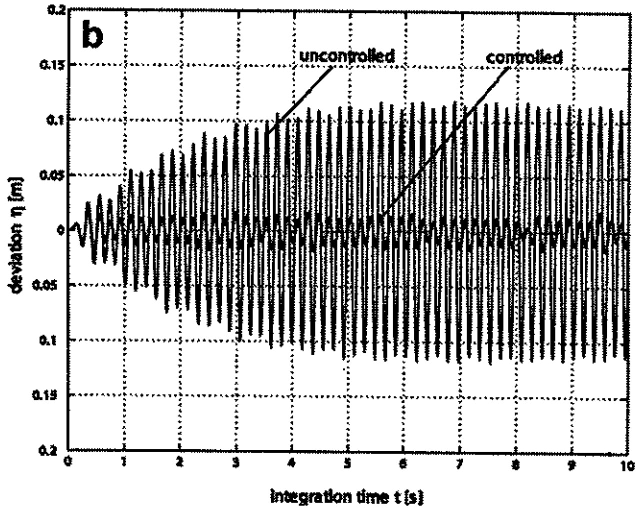


Fig. 4. a Deviation ξ vs. time t at $\omega = 22$ Hz, **b** deviation η vs. time t at $\omega = 22$ Hz

$\omega = 25$ Hz and $\omega = 33$ Hz, the significant improvement of $\dot{\eta}$ and $\ddot{\eta}$ prevails. In addition, Fig. 4 shows the deviation ξ and η vs. time t at some exemplary excitation frequency $\omega = 22$ Hz. Here, we see that a significant improvement of vibration attenuation holds not only for velocities and accelerations, but also for the deviations themselves, which are actually not the target of the proposed control.

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